

UNIQUENESS OF THE MOMENTUM MAP

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ABSTRACT. We give a detailed discussion of existence and uniqueness of Lu's momentum map. We introduce the infinitesimal momentum map, and analyze its integrability to the usual momentum map, its existence and its deformations.

CONTENTS

1. Introduction	1
2. Preliminaries: Poisson actions and Momentum maps	3
2.1. Dressing Transformations	4
3. The infinitesimal momentum map	5
3.1. The structure of a momentum map	5
3.2. The reconstruction problem	8
4. Infinitesimal deformations of a momentum map	10
References	12

1. INTRODUCTION

The classical momentum map for an action of a Lie group on a Poisson manifold provides a mathematical formalization of the notion of conserved quantity associated to symmetries of a dynamical system. The standard definition of momentum map only requires a canonical Lie algebra action and its existence is guaranteed whenever the infinitesimal generators of the Lie algebra action are Hamiltonian vector fields (modulo vanishing of a certain lie algebra cohomology class). In this paper we focus on a generalization of the momentum map provided by Lu [5], [6].

The detailed construction of this generalized momentum map and its basic properties are recalled in the following section. The basic structure is as follows. Given Poisson Lie group (G, π_G) one introduces the dual Poisson Lie group (G^*, π_{G^*}) and, under fairly general conditions, G^* carries a Poisson action of G (and vice versa). The Lie algebra \mathfrak{g} of G is naturally identified with the space of G^* -(left-)invariant one-forms on G^* :

$$\alpha : \mathfrak{g} \rightarrow \Omega^1(G^*)^{G^*},$$

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Given a Poisson manifold (M, π_M) with a Poisson action of G , a momentum map is a smooth, G -equivariant Poisson map

$$\mu : M \rightarrow G^*$$

satisfying

$$\mathcal{L}_\xi = \pi_M^\#(\mu^*(\alpha(\xi)))$$

where \mathcal{L} is the map $\mathfrak{g} \rightarrow \text{Vect}(M)$ induced by the action of G on M . A canonical example of a momentum map is the identity map $G^* \rightarrow G^*$, in which case α coincides with the structure one-form $\theta \in \Omega^1(G^*, \mathfrak{g}^*)^{G^*}$ of the Lie group G^* .

The Poisson structure on G gives its Lie algebra a structure of a Lie bialgebra $(\mathfrak{g}, [,], \delta)$ and hence a structure of Gerstenhaber algebra on $\Lambda^*\mathfrak{g}$. On the other hand the Poisson bracket on M gives $\Omega^*(M)[1]$ a structure of Lie algebra with bracket $[\cdot, \cdot]_{\pi_M}$ which induces a structure of Gerstenhaber algebra on $\Omega^*(M)$. The map α from above lifts to a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [,]) \longrightarrow (\Omega^\bullet(M), d_{DR}, [,]_\pi).$$

which we will call an *infinitesimal momentum map* (cf. the subsection 3.1 and the proposition 3). The main subject of this paper is the study of the properties of this infinitesimal momentum map and its relation to the usual momentum map. In particular, we show under which conditions it integrates to the usual momentum map.

The fact that α is map of Gerstenhaber algebras reduces to two equations

$$\alpha_{[\xi, \eta]} = [\alpha_\xi, \alpha_\eta]_\pi \text{ and } d\alpha_\xi + \frac{1}{2}\alpha \wedge \alpha \circ \delta(\xi) = 0$$

The second is a Maurer-Cartan type equation, in fact, in the case when $M = G^*$ it is precisely the Maurer Cartan equation for the Lie group G^* . In the case when Ω^\bullet is formal, the second equation admits explicit solution modulo gauge equivalence (cf. Theorem 3.2).

Theorem *Suppose that M is a Kähler manifold. The set of gauge equivalence classes of $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ satisfying the equation*

$$(1) \quad d\alpha_\xi + \frac{1}{2}\alpha \wedge \alpha \circ \delta(\xi) = 0$$

is in bijective correspondence with the set of the cohomology classes $c \in H^1(M, \mathfrak{g}^)$ satisfying*

$$(2) \quad [c, c] = 0.$$

The following describes conditions under which an infinitesimal momentum map integrates to the usual momentum map (cf. theorem 3.1 for the details).

Theorem *Let (M, π) be a Poisson manifold and $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$ an infinitesimal momentum map. Suppose that M and G are simply connected and G is compact. Then $\mathcal{D} = \{\alpha_\xi - \theta_\xi, \xi \in \mathfrak{g}\}$ generates an involutive distribution on $M \times G^*$ and a leaf $\mu_{\mathcal{F}}$ of \mathcal{D} is a graph of a momentum map if*

$$(3) \quad \pi(\alpha_\xi, \alpha_\eta) - \pi_{G^*}(\theta_\xi, \theta_\eta)|_{\mathcal{F}} = 0, \quad \xi, \eta \in \mathfrak{g}$$

In the section 3.2 we study concrete cases of this globalization question and prove the existence and uniqueness/nonuniqueness of a momentum map associated to a given infinitesimal momentum map for the particular case when the dual Poisson Lie group is abelian, respectively the Heisenberg group. For the second case the result is as follows (cf. Theorem 3.4).

Theorem *Let G be a Poisson Lie group acting on a Poisson manifold M with an infinitesimal momentum map α and such that G^* is the Heisenberg group. Let ξ, η, ζ denote the basis of \mathfrak{g} dual to the standard basis x, y, z of \mathfrak{g}^* , with z central and $[x, y] = z$. Then*

$$(4) \quad \pi(\alpha_\xi, \alpha_\eta) = c$$

where c is a constant on M . The form α lifts to a momentum map $\mu : M \rightarrow G^*$ if and only if $c = 0$. When $c = 0$ the set of momentum maps with given α is one dimensional with free transitive action of \mathbb{R} .

Finally, in the last section, we study the question of infinitesimal deformations of a given momentum map. The main result is Theorem 4.1, which describes explicitly the space tangent to the space of momentum maps at a given point. The main result can be formulated as a statement that the space of momentum maps has a structure of flat manifold (in an appropriate C^∞ topology).

Theorem *Infinitesimal deformations of a momentum map are given by smooth maps $X : M \rightarrow \mathfrak{g}^*$ satisfying the equations*

For all $\xi, \eta \in \mathfrak{g}$,

$$(5) \quad L_\xi X(\eta) - L_\eta X(\xi) = X([\xi, \eta])$$

$$(6) \quad \{X(\xi), \cdot\} = -L_{ad_X^* \xi}.$$

This theorem has the following corollary (cf Corollary 4.2).

Theorem *Suppose that G is a compact and semisimple Poisson Lie group with Poisson action on a Poisson manifold M and with a momentum map μ . Any smooth deformation of μ is given by integrating a Hamiltonian flow on M commuting with the action of G .*

2. PRELIMINARIES: POISSON ACTIONS AND MOMENTUM MAPS

In this section we give a brief summary of the notions of Poisson action and momentum map in the Poisson context. We discuss the dressing transformations as an example of Poisson actions that will allow us to introduce the concept of Hamiltonian action.

Recall that a Poisson Lie group (G, π_G) is a Lie group equipped with a multiplicative Poisson structure π_G . From the Drinfeld theorem [1], given a Poisson Lie group (G, π_G) , the linearization of π_G at e defines a Lie algebra structure on \mathfrak{g}^* such that $(\mathfrak{g}, \mathfrak{g}^*)$ form a Lie bialgebra over \mathfrak{g} . For this reason, in the following we always assume that G is connected and simply connected.

Definition 2.1. *The action of (G, π_G) on (M, π) is called **Poisson action** if the map $\Phi : G \times M \rightarrow M$ is Poisson, where $G \times M$ is a Poisson product with structure $\pi_G \oplus \pi$*

Given an action $\Phi : G \times M \rightarrow M$, we denote by L_ξ the Lie algebra anti-homomorphism from \mathfrak{g} to M which defines the infinitesimal generator of this action.

Proposition 1. *Assume that (G, π_G) is a connected Poisson Lie group. Then the action $\Phi : G \times M \rightarrow M$ is a Poisson action if and only if*

$$(7) \quad \mathcal{L}_{L_\xi}(\pi) = (\Phi \wedge \Phi)(\delta(\xi)),$$

for any $\xi \in \mathfrak{g}$, where $\delta = d_e \pi_G : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is the derivative of π_G at e .

The proof of this Proposition can be found in [7]. Motivated by this fact, we introduce the following definition.

Definition 2.2. *A Lie algebra action $\xi \mapsto L_\xi$ is called an **infinitesimal Poisson action** of the Lie bialgebra (\mathfrak{g}, δ) on (M, π) if it satisfies eq. (7).*

In this formalism the definition of momentum map reads (Lu, [5], [6]):

Definition 2.3. *A **momentum map** for the Poisson action $\Phi : G \times M \rightarrow M$ is a map $\mu : M \rightarrow G^*$ such that*

$$(8) \quad L_\xi = \pi^\sharp(\mu^*(\theta_\xi))$$

where θ_ξ is the left invariant 1-form on G^* defined by the element $\xi \in \mathfrak{g} = (T_e G^*)^*$ and μ^* is the cotangent lift $T^*G^* \rightarrow T^*M$.

2.1. Dressing Transformations. One of the most important example of Poisson action is the dressing action of G on G^* . Consider a Poisson Lie group (G, π_G) , its dual (G^*, π_{G^*}) and its double \mathcal{D} , with Lie algebras \mathfrak{g} , \mathfrak{g}^* and \mathfrak{d} , respectively.

Let $l(\xi)$ the vector field on G^* defined by

$$(9) \quad l(\xi) = \pi_{G^*}^\sharp(\theta_\xi)$$

for each $\xi \in \mathfrak{g}$. Here θ_ξ is the left invariant 1-form on G^* defined by $\xi \in \mathfrak{g} = (T_e G^*)^*$. The map $\xi \mapsto l(\xi)$ is a Lie algebra anti-homomorphism. Using the Maurer-Cartan equation for G^* :

$$(10) \quad d\theta_\xi + \frac{1}{2}\theta \wedge \theta \circ \delta(\xi) = 0$$

The action $\xi \mapsto l(\xi)$ is an infinitesimal Poisson action of the Lie bialgebra \mathfrak{g} on the Poisson Lie group G^* , called left infinitesimal dressing action (see f. ex. [5]).

Similarly, the right infinitesimal dressing action of \mathfrak{g} on G^* is defined by $r(\xi) = -\pi_{G^*}^\sharp(\theta_\xi)$ where θ_ξ is the right invariant 1-form on G^* .

Let $l(\xi)$ (resp. $r(\xi)$) a left (resp. right) dressing vector field on G^* . If all the dressing vector fields are complete, we can integrate the \mathfrak{g} -action into a Poisson G -action on G^* called the **dressing action** and we say that the dressing actions consist of dressing transformations. The orbits of the dressing actions are precisely the symplectic leaves in G (see [9], [5]).

The momentum map for the dressing action of G on G^* is the opposite of the identity map from G^* to itself.

Definition 2.4. A multiplicative Poisson tensor π on G is complete if each left (equiv. right) dressing vector field is complete on G .

It has been proved in [5] that a Poisson Lie group is complete if and only if its dual Poisson Lie group is complete.

Assume that G is a complete Poisson Lie group. We denote respectively the left (resp. right) dressing action of G on its dual G^* by $g \mapsto l_g$ (resp. $g \mapsto r_g$).

Definition 2.5. A momentum map $\mu : M \rightarrow G^*$ for a left (resp. right) Poisson action Φ is called **G -equivariant** if it is such with respect to the left dressing action of G on G^* , that is, $\mu \circ \Phi_g = \lambda_g \circ \mu$ (resp. $\mu \circ \Phi_g = \rho_g \circ \mu$)

A momentum map is G -equivariant if and only if it is a Poisson map, i.e. $\mu_*\pi = \pi_{G^*}$. Given this generalization of the concept of equivariance introduced for Lie group actions, it is natural to call **Hamiltonian action** a Poisson action induced by an equivariant momentum map.

3. THE INFINITESIMAL MOMENTUM MAP

In this section we study the conditions for the existence and the uniqueness of the momentum map. In particular, we give a new definition of the momentum map, called infinitesimal, in terms of one-forms and we study the conditions under which the infinitesimal momentum map determines a momentum map in the usual sense. We describe the theory of reconstruction of the momentum map from the infinitesimal one in two explicit cases. Finally, we provide the conditions which ensure the uniqueness of the momentum map.

3.1. The structure of a momentum map. Recall that, for the Poisson Lie group G^* we identify \mathfrak{g} with the space of left invariant 1-forms on G^* ; this space is closed under the bracket defined by π_{G^*} and the induced bracket on \mathfrak{g} , by the above identification, coincides with the original Lie bracket on \mathfrak{g} (see [10]).

Proposition 2. Let θ_ξ, θ_η be two left invariant 1-forms on G^* , such that $\theta_\xi(e) = \xi$, $\theta_\eta(e) = \eta$ then

$$(11) \quad \theta_{[\xi, \eta]} = [\theta_\xi, \theta_\eta]_{\pi_{G^*}}$$

and

$$(12) \quad \mathcal{L}_X \pi_{G^*}(\theta_\xi, \theta_\eta) = x([\xi, \eta]) + \pi_{G^*}(\theta_{ad_x^* \xi}, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_{ad_x^* \eta})$$

Proof. Let us consider an element $x \in \mathfrak{g}^*$ and the correspondent left invariant vector field $X \in TG^*$. Recall that given a Poisson manifold, the Poisson structure always induces a Lie bracket on the space of one-form on the manifold (see [8]) by

$$(13) \quad [\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

Using this explicit formula for $[\theta_\xi, \theta_\eta]_{\pi_{G^*}}$ we can see that

$$(14) \quad \iota_X [\theta_\xi, \theta_\eta]_{\pi_{G^*}} = (\mathcal{L}_X \pi_{G^*})(\theta_\xi, \theta_\eta).$$

This proves that $[\theta_\xi, \theta_\eta]_{\pi_{G^*}}$ is a left invariant 1-form. In particular, since $\mathcal{L}_X \pi_{G^*}(e) = {}^t\delta(x)$, eq. (11) is proved¹.

¹This relation has already been claimed in [4]

Moreover, we have

$$(15) \quad \begin{aligned} \mathcal{L}_X \pi_{G^*}(\theta_\xi, \theta_\eta) &= (\mathcal{L}_X \pi_{G^*})(\theta_\xi, \theta_\eta) + \pi_{G^*}(\mathcal{L}_X \theta_\xi, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_\eta) \\ &= {}^t\delta(x)(\xi, \eta) + \pi_{G^*}(\theta_{ad_x^* \xi}, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_{ad_x^* \eta}), \end{aligned}$$

since $\mathcal{L}_X \theta_\xi = \theta_{ad_x^* \xi}$. From ${}^t\delta(x)(\xi, \eta) = x([\xi, \eta])$, eq. (12) follows. \square

As a direct consequence, recalling that the pullback and the differential commute and using the equivariance of the momentum map, we have the following proposition:

Proposition 3. *Given a Poisson action $\Phi : G \times M \rightarrow M$ with equivariant momentum map $\mu : M \rightarrow G^*$, the forms $\alpha_\xi = \mu^*(\theta_\xi)$ satisfy the following identities:*

$$(16) \quad \alpha_{[\xi, \eta]} = [\alpha_\xi, \alpha_\eta]_\pi$$

$$(17) \quad d\alpha_\xi + \frac{1}{2}\alpha \wedge \alpha \circ \delta(\xi) = 0$$

This motivates the following Definition.

Definition 3.1. *Let M be a Poisson manifold and G a Poisson Lie group. An **infinitesimal momentum map** is a morphism of Gerstenhaber algebras*

$$(18) \quad \alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\cdot, \cdot]) \longrightarrow (\Omega^\bullet(M), d_{DR}, [\cdot, \cdot]_\pi).$$

The following theorem describes the conditions in which an infinitesimal momentum map determines a momentum map in the usual sense.

Theorem 3.1. *Let (M, π) be a Poisson manifold and $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$ a linear map which satisfies the conditions (16)-(17). Then:*

- (i) *The set $\{\alpha_\xi - \theta_\xi, \xi \in \mathfrak{g}\}$ generate an involutive distribution \mathcal{D} on $M \times G^*$.*
- (ii) *If M is connected and simply connected, the leaves \mathcal{F} of \mathcal{D} coincide with the graphs of the maps $\mu_{\mathcal{F}} : M \rightarrow G^*$ satisfying $\alpha = \mu_{\mathcal{F}}^*(\theta)$ and G^* acts freely and transitively on the space of leaves by left multiplication on the second factor.*
- (iii) *The vector fields $\pi^\sharp(\alpha_\xi)$ define a homomorphism from \mathfrak{g} to TM . If they integrate to the action $\Phi : G \times M \rightarrow M$ (e.g. when M is compact and G simply connected), then Φ is a Poisson action and $\mu_{\mathcal{F}}$ is its momentum map if and only if the functions*

$$(19) \quad \varphi(\xi, \eta) = \pi(\alpha_\xi, \alpha_\eta) - \pi_{G^*}(\theta_\xi, \theta_\eta)$$

satisfy

$$(20) \quad \varphi(\xi, \eta)|_{\mathcal{F}} = 0$$

for all $\xi, \eta \in \mathfrak{g}$.

Proof. (i) Using the eqs. (10) and (17), the \mathfrak{g} -valued form $\alpha - \theta$ on $M \times G^*$ satisfies $d(\alpha - \theta) = (\alpha - \theta) \wedge (\alpha - \theta)$; as a consequence, from the Frobenius theorem, it defines a distribution on $M \times G^*$. Let \mathcal{F} be any of its leaves and let $p_i, i = 1, 2$ denote the projection onto the first (resp. second) factor in $M \times G^*$. Since the linear span of $\theta_\xi, \xi \in \mathfrak{g}$ at any point $u \in G^*$ coincides with $T_u^* G^*$, the restriction of the projection $p_1 : M \times G^* \rightarrow M$ to \mathcal{F} is an immersion. Finally, since $\dim(M) = \dim(\mathcal{F})$, p_1 is a covering map.

- (ii) Under the hypothesis that M is simply connected, p_1 is a diffeomorphism and

$$(21) \quad \mu_{\mathcal{F}} = p_2 \circ p_1^{-1}$$

is a smooth map whose graph coincides with \mathcal{F} . It is immediate, that $\alpha = \mu_{\mathcal{F}}^*(\theta)$. Moreover, since θ 's are left invariant it follows immediately that the action of G^* on the space of leaves by left multiplication of the second factor is free and transitive.

- (iii) Suppose that the condition (20) is satisfied. Then

$$(22) \quad \pi(\alpha_{\xi}, \alpha_{\eta}) = \mu_{\mathcal{F}}^*(\pi_{G^*}(\theta_{\xi}, \theta_{\eta}))$$

and $\text{Ker} \mu_{\mathcal{F}*}$ coincides with the set of zero's of α_{ξ} , $\xi \in \mathfrak{g}$. Hence, $\mu_{\mathcal{F}}$ is a Poisson map and, in particular

$$(23) \quad \mu_{\mathcal{F}*}(\pi^{\sharp}(\alpha_{\xi})) = \pi_{G^*}^{\sharp}(\theta_{\xi}),$$

i.e. it is a G -equivariant map.

□

The first of the equations in the definition 3, the equation

$$(24) \quad d\alpha_{\xi} + \frac{1}{2}\alpha \wedge \alpha \circ \delta(\xi) = 0,$$

can be solved explicitly in the case when M is a Kähler manifold. Before stating the result, we need to introduce the concept of gauge equivalence of solutions of (24):

Definition 3.2. Two solutions α and α' of eq. (24) are said to be ***gauge equivalent***, if there exists a smooth function $\mathcal{H} : M \rightarrow \mathfrak{g}^*$ such that

$$(25) \quad \alpha' = \exp(ad\mathcal{H})(\alpha) + \int_0^1 dt \exp t(ad\mathcal{H})(d\mathcal{H})$$

Theorem 3.2. Suppose that M is a Kähler manifold. The set of gauge equivalence classes of $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ satisfying the equation

$$(26) \quad d\alpha_{\xi} + \frac{1}{2}\alpha \wedge \alpha \circ \delta(\xi) = 0$$

is in bijective correspondence with the set of the cohomology classes $c \in H^1(M, \mathfrak{g}^*)$ satisfying

$$(27) \quad [c, c] = 0.$$

Proof. Since M is a Kähler manifold, $(\Omega^{\bullet}(M), d)$ is a formal CDGA (commutative differential graded algebra) [3]. As a consequence,

$$(28) \quad \text{Hom}(\mathfrak{g}^*, \Omega^{\bullet}(M)), d, [\cdot, \cdot]$$

is a formal DGLA (some elements of DGLA's will be given in the next chapter) and, in particular, there exists a bijection between the equivalence classes of Maurer Cartan elements of $\text{Hom}(\mathfrak{g}^*, \Omega^{\bullet}(M), d, [\cdot, \cdot])$ and Maurer Cartan elements of $\text{Hom}(\mathfrak{g}^*, H_{dR}^{\bullet}(M), [\cdot, \cdot])$.

A Maurer-Cartan element in $\text{Hom}(\mathfrak{g}^*, H_{dR}^{\bullet}(M), [\cdot, \cdot])$ is an element c in $H^1(M, \mathfrak{g}^*)$ satisfying

$$(29) \quad [c, c] = 0,$$

and the claim is proved. \square

3.2. The reconstruction problem. In this section we discuss the conditions under which the distribution \mathcal{D} defined in Theorem 3.1 admits a leaf satisfying eq. (20). In particular, we analyze the case where the structure on G^* is trivial and the Heisenberg group case. In the following we keep the assumption that M is connected and simply connected.

3.2.1. The abelian case. Suppose that $G^* = \mathfrak{g}^*$ is abelian. Then, the forms α_ξ satisfy $d\alpha_\xi = 0$, hence $\alpha_\xi = dH_\xi$ (since $H^1(M) = 0$), for some $H_\xi \in C^\infty(M)$.

Let us denote by ev_ξ the linear functions $\mathfrak{g}^* \ni l \rightarrow z(\xi)$. Then $\theta_\xi = d(ev_\xi)$ and the leaves of the distribution \mathcal{D} coincide with the level sets (on $M \times \mathfrak{g}^*$) of the functions

$$(30) \quad \{H_\xi - ev_\xi \mid \xi \in \mathfrak{g}\}.$$

Furthermore, we have

$$(31) \quad \varphi(\xi, \eta)(m, z) = \{H_\xi, H_\eta\} - z([\xi, \eta]).$$

In this case, the basic identity (13) reduces to

$$(32) \quad d\{H_\xi, H_\eta\} = dH_{[\xi, \eta]},$$

hence

$$(33) \quad \{H_\xi, H_\eta\} - H_{[\xi, \eta]} = c(\xi, \eta),$$

for some constants $c(\xi, \eta)$. By the Jacobi identity, the constants $c(\xi, \eta)$ define a class $[c] \in H^2(\mathfrak{g}, \mathbb{R})$. Suppose that this class vanishes (for example if \mathfrak{g} semisimple). Then, there exists a $z_0 \in \mathfrak{g}^*$ such that $c(\xi, \eta) = z_0([\xi, \eta])$. Hence, given a leaf \mathcal{F} ,

$$(34) \quad \varphi(\xi, \eta)|_{\mathcal{F}} = 0$$

if and only if \mathcal{F} is given by

$$(35) \quad H_\xi - ev_\xi - z_0(\xi) = 0.$$

In other words, the space of leaves of \mathcal{D} which give a momentum map coincides with the affine space modeled on $\{z \in \mathfrak{g}^* : z|_{[\mathfrak{g}, \mathfrak{g}]} = 0\}$ (which again vanishes when \mathfrak{g} is semisimple). This proves the following theorem.

Theorem 3.3. *Suppose that G is a connected and simply connected Lie group with trivial Poisson structure and M is compact. Then an infinitesimal momentum map is a map $H : \mathfrak{g} \rightarrow C^\infty(M) : \xi \mapsto H_\xi$ such that*

$$(36) \quad d\{H_\xi, H_\eta\} = dH_{[\xi, \eta]}, \quad \forall \xi, \eta \in \mathfrak{g}.$$

The element $c(\xi, \eta) = \{H_\xi, H_\eta\} - H_{[\xi, \eta]}$ is a two cocycle c on \mathfrak{g} with values in \mathbb{R} . The infinitesimal momentum map H is generated by a momentum map μ if this cocycle vanishes and, in this case, μ is unique.

3.2.2. *the Heisenberg group case.* Suppose now that G^* is the Heisenberg group. Let x, y, z be a basis for \mathfrak{g}^* , where z is central and $[x, y] = z$. Let ξ, η, ζ be the dual basis of \mathfrak{g} . The cocycle δ on \mathfrak{g} is given by

$$(37) \quad \delta(\xi) = \delta(\eta) = 0 \quad \text{and} \quad \delta(\zeta) = \xi \wedge \eta,$$

then

$$(38) \quad \begin{aligned} d\alpha_\xi &= d\alpha_\eta = 0 \\ d\alpha_\zeta &= \alpha_\xi \wedge \alpha_\eta \end{aligned}$$

There are essentially two possibilities for the Lie bialgebra structure on \mathfrak{g}^* , which give the following two possibilities for the Lie algebra structure on \mathfrak{g} . Either

$$(39) \quad [\xi, \eta] = 0, [\xi, \zeta] = \xi, [\eta, \zeta] = \eta$$

or

$$(40) \quad [\xi, \eta] = 0, [\xi, \zeta] = \eta, [\eta, \zeta] = -\xi.$$

The result below will turn out to be independent of the choice (the computations will be done using the second choice, which corresponds to $G = \mathbb{R} \ltimes \mathbb{R}^2$, with \mathbb{R} acting by rotation on \mathbb{R}^2). Below we use the notation

$$(41) \quad \delta(\xi) = \sum_i \xi_i^1 \wedge \xi_i^2.$$

Applying the Cartan formula $\mathcal{L} = [\iota, d]$ and the identity $[\alpha_\xi, \alpha_\eta]_\pi = \alpha_{[\xi, \eta]}$ to the basic equation (13), we get

$$(42) \quad \sum_i \pi(\alpha_\eta, \alpha_{\xi_i^1}) \alpha_{\xi_i^2} - \sum_i \pi(\alpha_\xi, \alpha_{\eta_i^1}) \alpha_{\eta_i^2} = \alpha_{[\eta, \xi]} - d\pi(\alpha_\eta, \alpha_\xi).$$

In our case it gives the following equations

$$(43) \quad \begin{aligned} d\pi(\alpha_\xi, \alpha_\eta) &= \alpha_{[\xi, \eta]} \\ d\pi(\alpha_\zeta, \alpha_\eta) &= \alpha_{[\zeta, \eta]} + \pi(\alpha_\eta, \alpha_\xi) \alpha_\eta \\ d\pi(\alpha_\zeta, \alpha_\xi) &= \alpha_{[\zeta, \xi]} - \pi(\alpha_\xi, \alpha_\eta) \alpha_\xi \end{aligned}$$

which are also satisfied after replacing α with θ . Let \mathcal{I} denote the ideal generating our distribution \mathcal{D} . Then, from above,

$$(44) \quad d\varphi(\xi, \eta) \in \mathcal{I}$$

and

$$(45) \quad \varphi(\xi, \eta)|_{\mathcal{F}} = 0 \implies d\varphi(\zeta, \eta)|_{\mathcal{F}} \text{ and } d\varphi(\zeta, \xi)|_{\mathcal{F}} \in \mathcal{I}.$$

Here, as before, \mathcal{F} is a leaf of \mathcal{D} . Using the relation (12), we get

$$(46) \quad \begin{aligned} \mathcal{L}_z^*(\pi_{G^*}(\theta_\xi, \theta_\eta)) &= \mathcal{L}_x^*(\pi_{G^*}(\theta_\xi, \theta_\eta)) = \mathcal{L}_y^*(\pi_{G^*}(\theta_\xi, \theta_\eta)) = 0 \\ \mathcal{L}_z^*(\pi_{G^*}(\theta_\xi, \theta_\zeta)) &= \mathcal{L}_y^*(\pi_{G^*}(\theta_\xi, \theta_\zeta)) = 0 \\ \mathcal{L}_z^*(\pi_{G^*}(\theta_\eta, \theta_\zeta)) &= \mathcal{L}_x^*(\pi_{G^*}(\theta_\eta, \theta_\zeta)) = 0 \\ \mathcal{L}_x^*(\pi_{G^*}(\theta_\xi, \theta_\zeta)) &= 1 \\ \mathcal{L}_y^*(\pi_{G^*}(\theta_\eta, \theta_\zeta)) &= 1 \end{aligned}$$

In particular, $\pi_{G^*}(\theta_\xi, \theta_\eta)$ is invariant under left translations. Since π_{G^*} is zero at the identity, we get

$$(47) \quad \pi_{G^*}(\theta_\xi, \theta_\eta) = 0$$

Since $d\varphi(\xi, \eta) \in \mathcal{I}$, the function $\varphi(\xi, \eta)$ is leafwise constant. Using the definition (19) and the equation (47) it follows that also $\pi(\alpha_\xi, \alpha_\eta)$ is leafwise constant. Hence we have

Lemma 3.1. *$\pi(\alpha_\xi, \alpha_\eta) = c$ is constant on M and necessary condition for existence of the momentum map is $c = 0$.*

Let us continue under the assumption that $c = 0$. Then, given a leaf \mathcal{F} , by eq. (44),

$$(48) \quad \varphi(\eta, \zeta)|_{\mathcal{F}} = c_1 \quad \text{and} \quad \varphi(\xi, \zeta)|_{\mathcal{F}} = c_2$$

for some constants c_1 and c_2 . Setting $\mathcal{F}_1 = id \times \exp(c_1 x) \exp(c_2 y)$ to \mathcal{F} , we get

$$(49) \quad \varphi(\eta, \zeta)|_{\mathcal{F}_1} = \varphi(\xi, \zeta)|_{\mathcal{F}_1} = \varphi(\xi, \eta)|_{\mathcal{F}_1} = 0.$$

The final result is as follows.

Theorem 3.4. *Let G be a Poisson Lie group acting on a Poisson manifold M with an infinitesimal momentum map α and such that G^* is the Heisenberg group. Let ξ, η, ζ denote the basis of \mathfrak{g} dual to the standard basis x, y, z of \mathfrak{g}^* , with z central and $[x, y] = z$. Then*

$$(50) \quad \pi(\alpha_\xi, \alpha_\eta) = c$$

where c is a constant on M . The form α lifts to a momentum map $\mu : M \rightarrow G^*$ if and only if $c = 0$. When $c = 0$ the set of momentum maps with given α is one dimensional with free transitive action of \mathbb{R} .

4. INFINITESIMAL DEFORMATIONS OF A MOMENTUM MAP

In the following we study infinitesimal deformations of a given momentum map.

Let (M, π) be a Poisson manifold with a Poisson action of a Poisson Lie group (G, π_G) generated by the momentum map $\mu : M \rightarrow G^*$. In the following we denote by $\exp : \mathfrak{g}^* \rightarrow G^*$ the exponential map. Suppose that $[-\epsilon, \epsilon] \ni t \rightarrow \mu_t : M \rightarrow G^*$, $\epsilon > 0$, is a differentiable path of momentum maps for this action. We can assume that $\mu_t(m)$ is of the form

$$(51) \quad \mu(m) \exp(tX_m + t^2 Y(t, m))$$

for some differentiable maps $X : M \rightarrow \mathfrak{g}^* : m \mapsto X(m)$ and $Y :]-\epsilon, \epsilon[\times M \rightarrow G^*$.

Theorem 4.1. *In the notation above the following identities hold.*

For all $\xi, \eta \in \mathfrak{g}$,

$$(52) \quad L_\xi X(\eta) - L_\eta X(\xi) = X([\xi, \eta])$$

$$(53) \quad \{X(\xi), \cdot\} = -L_{ad_X^* \xi}.$$

Proof. Let us compute

$$\beta_\xi = \left. \frac{d}{dt} \right|_{t=0} \langle d\mu_t, \theta_\xi \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle d(\mu \exp(tX)), \theta_\xi \rangle$$

. First note that

$$(54) \quad d(\mu \exp(tX)) = (\rho_{\exp(tX)})_* d\mu + (\lambda_\mu)_* d\exp(tX),$$

where ρ and λ are the right and left multiplication, respectively. Calculating the derivative $\frac{d}{dt}\Big|_{t=0}$ we get:

$$(55) \quad \begin{aligned} \frac{d}{dt}\Big|_{t=0} \langle (\rho_{\exp(tX)})_* d\boldsymbol{\mu}, \theta_\xi \rangle &= \frac{d}{dt}\Big|_{t=0} \langle d\boldsymbol{\mu}, (\rho_{\exp(tX)})^* \theta_\xi \rangle \\ &= \langle d\boldsymbol{\mu}, \mathcal{L}_X \theta_\xi \rangle \\ &= \langle d\boldsymbol{\mu}, \theta_{ad_X^* \xi} \rangle = \alpha_{ad_X^* \xi} \end{aligned}$$

and

$$(56) \quad \begin{aligned} \frac{d}{dt}\Big|_{t=0} \langle (\lambda_\mu)_* d\exp(tX), \theta_\xi \rangle &= \frac{d}{dt}\Big|_{t=0} \langle d\exp(tX), (\lambda_\mu)^* \theta_\xi \rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle d\exp(tX), \theta_\xi \rangle \end{aligned}$$

The differential of the exponential map $\exp : \mathfrak{g}^* \rightarrow G^*$ is a map from the cotangent bundle of \mathfrak{g}^* to the cotangent bundle of G^* . It can be trivialized as $d\exp : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow G^* \times \mathfrak{g}^*$. Furthermore, $(\exp^{-1}, id) : G^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ hence the map $\mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ is given by $tX + o(t^2)$. We get

$$(57) \quad \frac{d}{dt}\Big|_{t=0} \langle d\exp(tX), \theta_\xi \rangle = \frac{d}{dt}\Big|_{t=0} \langle d(tX + o(t)), \theta_\xi \rangle = d\langle X, \theta_\xi \rangle = d\langle X, \xi \rangle$$

and finally

$$(58) \quad \beta_\xi = \alpha_{ad_X^* \xi} + dX(\xi).$$

Since $\pi^\sharp(\alpha_\xi^t) = L_\xi$ is independent of t , $\pi^\sharp\beta_\xi = 0$ and we get the identity (53).

In order to prove the relation (52), recall that, since $\boldsymbol{\mu}_t$ is a family of Poisson maps, one has

$$(59) \quad \pi(\alpha_\xi^t, \alpha_\eta^t)(m) = \pi_{G^*}(\theta_\xi, \theta_\eta)(\boldsymbol{\mu}_t(m)).$$

Applying $\frac{d}{dt}\Big|_{t=0}$ to both sides, we get

$$(60) \quad \pi(\beta_\xi, \alpha_\eta)(m) + \pi(\alpha_\xi, \beta_\eta)(m) = \mathcal{L}_X(\pi_{G^*}(\theta_\xi, \theta_\eta))(\boldsymbol{\mu}(m)).$$

Substituting the expression of β 's (58) and using the following identity

$$(61) \quad \mathcal{L}_X(\pi_{G^*}(\theta_\xi, \theta_\eta))(\boldsymbol{\mu}(m)) = X([\xi, \eta]) + \pi_{G^*}(\theta_{ad_X^* \xi}, \theta_\eta) + \pi_{G^*}(\theta_\xi, \theta_{ad_X^* \eta})$$

the claimed equality follows. \square

Corollary 4.2. *Suppose that M is a Poisson manifold with a poisson action of a compact semisimple Poisson Lie group G . Then any infinitesimal deformation of a momentum map $\boldsymbol{\mu} : M \rightarrow G^*$ as above is generated by a one parameter family of gauge transformations.*

Proof. Since the relation (52) implies that $X \in H^1(\mathfrak{g}, C^\infty(M, \mathfrak{g}^*))$ and G is compact semisimple, X is a Lie coboundary, i. e. there exists a function

$$\Phi : M \rightarrow \mathfrak{g}^*$$

such that

$$(62) \quad L_\xi \Phi = X(\xi).$$

In particular, it is easy to check that $L_{ad_X^*} \xi f = \sum L_{\xi'_i} \Phi \xi''_i(f)$, where $\delta(\xi) = \sum \xi'_i \otimes \xi''_i$. Now observe that

$$(63) \quad \xi\{\Phi, f\} = L_\xi \pi(d\Phi, df) = (L_\xi \pi)(d\Phi, df) + \{L_\xi \Phi, f\} + \{\Phi, L_\xi f\}$$

hence

$$(64) \quad \begin{aligned} \{X(\xi), f\} &= \{L_\xi \Phi, f\} = \xi\{\Phi, f\} - (L_\xi \pi)(d\Phi, df) - \{\Phi, L_\xi f\} \\ &= \xi\{\Phi, f\} - \delta(\xi)(\Phi, f) - \{\Phi, L_\xi f\} \\ &= \xi\{\Phi, f\} - L_{\xi'} \Phi \xi''(f) - \{\Phi, L_\xi f\}. \end{aligned}$$

Substituting the equations 62 and 64 in (53) we get

$$(65) \quad \xi\{\Phi, f\} - \{\Phi, L_\xi f\} = 0.$$

In other words the hamiltonian vector field associated to Φ commutes with the group action and is tangent to the derivative of μ_t at $t = 0$ as claimed. \square

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